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A NEW TRIANGULATION OF THE UNIT SIMPLEX FOR COMPUTING ECONOMIC EQUILIBRIA

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ABSTRACT

In order to compute economic equilibria in an exchange economy by means of simplicial algorithms, we propose a new simplicial subdivision of the underlying price space which is the unit simplex. We call this simplicial subdivision of the unit simplex the T_1'-triangulation. It is showed that the T_1'-triangulation is superior to the other well-known triangulations of the unit simplex according to measures of efficiency of triangulations.

1. INTRODUCTION

Since van der Laan and Talman proposed the first simplicial variable dimension algorithm on the unit simplex without an extra dimension in [8], a lot of contributions to this field have appeared, for example, see [3], [4], [6], [8], and [11]. These algorithms can be used to find an economic equilibrium in a pure exchange economy. They subdivide the price space, which is the n-dimensional unit simplex S^n, into simplices and search for a simplex that yields an approximate equilibrium. So far, we might say that the simplicial subdivisions, which underly all these algorithms, are adaptations or generalizations to S^n of the well-known K_1-triangulation and J_1-triangulation of R^n. In [2], we proposed a new triangulation of R^n, called the D_1-triangulation. It is superior to the other well-known triangulations of R^n. In order to improve simplicial algorithms on the unit simplex, we want to construct a new simplicial subdivision of the unit simplex, based on the D_1-triangulation.

Section 2 introduces the new triangulation of the unit simplex, the T_1'-triangulation. Its pivot rules are given in Section 3. The comparisons of triangulations of the unit simplex are presented in Section 4.

2. A NEW TRIANGULATION OF THE UNIT SIMPLEX

Assume $n \geq 2$. Let $N = \{1, 2, \ldots, n\}$. Let m be a positive integer. Let

$$C^n(m) = \{x \in R^n \mid m \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$$

and let S^n denote the n-dimensional unit simplex,

$$S^n = \left\{ x \in R^{n+1} \mid \sum_{j=1}^{n+1} x_j = 1 \text{ and } x_j \geq 0 \text{ for } j = 1, 2, \ldots, n+1 \right\}.$$

We will construct a simplicial subdivision of $C^n(m)$. Then we get a simplicial subdivision of the unit simplex S^n through a linear transformation by translating all simplices of the triangulation of $C^n(m)$.

Let

$$T_1^c = \{y \in C^n(m) \mid \text{all components of } y \text{ are even}\}.$$

Let $y_0 = m$ and $y_{n+1} = 0$. Take $y \in T_1^c$. Define

$$I(y) = \{i \in N \mid y_{i+1} < y_i < y_{i-1}\},$$

$$I^+(y) = \{i \in N \mid y_{i+1} = y_i < y_{i-1}\},$$

$$I^-(y) = \{i \in N \mid y_{i+1} < y_i = y_{i-1}\},$$

and

$$J(y) = \{i \in N \mid y_{i+1} = y_i = y_{i-1}\}.$$

Then it is obvious that for each $y \in T_1^c$,

$$N = I(y) \cup I^+(y) \cup I^-(y) \cup J(y).$$

Take $s = (s_1, s_2, \ldots, s_n)^\top$ to be a sign vector such that $s_i \in \{-1, +1\}$ for $i = 1, 2, \ldots, n$ and such that for $j \in N \backslash I(y)$, if $s_j = 1$ then $s_k = 1$ for all $k < j$ with $y_k = y_j$, if $s_j = -1$ then $s_k = -1$ for all $k > j$ with $y_k = y_j$, and if $y_1 = m$ then $s_1 = -1$, and if $y_n = 0$ then $s_n = 1$.

Let $K(y, s)$ denote the set

$$K(y, s) = \left\{ i \mid i \in I(y), \text{or } i \in I^+(y) \text{ and } s_i = 1, \text{or } i \in I^-(y) \text{ and } s_i = -1 \right\}.$$

Take a permutation $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ of the elements of N such that for $k, j \in N \backslash I(y)$ with $k < j$ and $y_j = y_k$, if $s_j = 1$ then $\pi^{-1}(k) > \pi^{-1}(j)$ and if $s_k = -1$ then $\pi^{-1}(k) < \pi^{-1}(j)$.

Let q denote the nonnegative integer such that $\pi(i) \in K(y,s)$ for $i = n-q+1, \ldots, n$ and $\pi(n-q) \notin K(y,s)$. Take an integer p such that $0 \leq p \leq q-1$.

Let u^h be the h-th unit vector in R^n for $h = 1, 2, \ldots, n$.

Definition 2.1. Let the vector y, the permutation π, the sign vector s, and the number p be given as above.

When $q < 2$, let $y^0 = y + s$,

$$y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)}, k = 1, 2, \ldots, n.$$

When $2 \leq q < n$, let $y^0 = y + s$,

$$y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)}, k = 1, 2, \ldots, n-q-1,$$

if $p = 0$, let $y^{n-q} = y$,

$$y^k = y + s_{\pi(k)} u^{\pi(k)}, k = n-q+1, \ldots, n;$$

if $p \geq 1$, let

$$y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)}, k = n-q, \ldots, n-q+p-1,$$
$$y^k = y + s_{\pi(k)} u^{\pi(k)}, k = n-q+p, \ldots, n.$$

When $q = n$, if $p = 0$, let $y^0 = y$,

$$y^k = y + s_{\pi(k)} u^{\pi(k)}, k = 1, 2, \ldots, n;$$

if $p \geq 1$, let $y^0 = y + s$,

$$y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)}, k = 1, 2, \ldots, p-1,$$
$$y^k = y + s_{\pi(k)} u^{\pi(k)}, k = p, p+1, \ldots, n.$$

Let $y^0, y^1, \ldots, y^n$ be produced in the above manner. Then it is obvious that they are affinely independent. Thus their convex hull is a simplex with vertices $y^0, y^1, \ldots, y^n$. Let us denote this simplex by $T_1(y, \pi, s, p)$. Let T_1 denote the set of all such simplices $T_1(y, \pi, s, p)$. Below we show that T_1 is a triangulation of $C^n(m)$.

Lemma 2.2. The union of all simplices in T_1 is equal to $C^n(m)$.

Proof. Clearly, every simplex in T_1 is contained in $C^n(m)$. Now let $x \in C^n(m)$ be arbitrary. Then $x \in T_1(y, \pi, s, p)$ with y, π, s, and p determined as follows.

The vector y is equal to

$$y_i = \begin{cases} \lfloor x_i \rfloor + 1 & \text{if } \lfloor x_i \rfloor \text{ is odd and } \lfloor x_i \rfloor < m, \\ \lfloor x_i \rfloor - 1 & \text{if } \lfloor x_i \rfloor \text{ is odd and } \lfloor x_i \rfloor = m, \\ \lfloor x_i \rfloor & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \dots, n$, and the sign vector s is equal to

$$s_i = \begin{cases} -1 & \text{if } \lfloor x_i \rfloor \text{ is odd and } \lfloor x_i \rfloor < m, \text{ or } \lfloor x_i \rfloor \text{ is even and } \lfloor x_i \rfloor = m, \\ 1 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \dots, n$. It is obvious that $y \in T_1^c$.

The permutation π is such that

$$s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}) \le s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) \le \cdots \le s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)})$$

and for $i < j$ with $s_{\pi(i)} = s_{\pi(j)}$ and $y_{\pi(i)} = y_{\pi(j)}$, if $s_{\pi(i)} = -1$ then $\pi(i) < \pi(j)$ and if $s_{\pi(i)} = 1$ then $\pi(i) > \pi(j)$.

When $q < 2$, then p is equal to 0. Let

$$\begin{aligned} \beta_0 &= s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}), \\ \beta_1 &= s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) - s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}), \\ &\dots \\ \beta_n &= s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)}) - s_{\pi(n-1)}(x_{\pi(n-1)} - y_{\pi(n-1)}). \end{aligned}$$

Then it is obvious that $\beta_k \ge 0$ for all k and that

$$\sum_{k=0}^{n} \beta_k = 1 \text{ and } x = \sum_{k=0}^{n} \beta_k y^k,$$

where y^k is as defined in Definition 2.1 for $k = 0, 1, \dots, n$. Thus

$$x \in T_1(y, \pi, s, p).$$

When $q = n$, the proof is the same as that of Lemma 2.2 in [2].

Suppose $2 \le q < n$. Let

$$\mu = -q s_{\pi(n-q)}(x_{\pi(n-q)} - y_{\pi(n-q)}) + \sum_{k=n-q}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}).$$

If $\mu \le 1$, then p is equal to 0. Let

$$\begin{aligned}
\beta_0 &= s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}),\\
\beta_1 &= s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) - s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}),\\
&\dots\\
\beta_{n-q-1} &= s_{\pi(n-q)}(x_{\pi(n-q)} - y_{\pi(n-q)}) - s_{\pi(n-q-1)}(x_{\pi(n-q-1)} - y_{\pi(n-q-1)}),\\
\beta_{n-q} &= 1 - \mu,\\
\beta_j &= s_{\pi(j)}(x_{\pi(j)} - y_{\pi(j)}) - s_{\pi(n-q)}(x_{\pi(n-q)} - y_{\pi(n-q)}),
\end{aligned}$$

for $j = n-q+1, \dots, n$. Then it is obvious that $\beta_k \ge 0$ for all k and that

$$\sum_{k=0}^{n} \beta_k = 1 \text{ and } x = \sum_{k=0}^{n} \beta_k y^k,$$

where y^k is as defined in Definition 2.1 for $k = 0, 1, \dots, n$. Thus

$$x \in T_1(y, \pi, s, p).$$

Now suppose $\mu > 1$. We show that there exists $1 \le p \le q-1$ such that the following system has a nonnegative solution,

$$\begin{aligned}
\beta_0 &= s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}),\\
\beta_1 &= s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) - s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}),\\
&\dots\\
\beta_{n-q+p-2} &= s_{\pi(n-q+p-1)}(x_{\pi(n-q+p-1)} - y_{\pi(n-q+p-1)})\\
&\quad - s_{\pi(n-q+p-2)}(x_{\pi(n-q+p-2)} - y_{\pi(n-q+p-2)}),\\
\beta_{n-q+p-1} &= s_{\pi(n-q+p-1)}(x_{\pi(n-q+p-1)} - y_{\pi(n-q+p-1)})\\
&\quad + (\textstyle\sum_{k=n-q+p}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - 1)/(q-p),\\
\beta_j &= s_{\pi(j)}(x_{\pi(j)} - y_{\pi(j)})\\
&\quad + (1 - \textstyle\sum_{k=n-q+p}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}))/(q-p),
\end{aligned}$$

for $j = n-q+p, \dots, n$.

If $\beta_{n-q+p-1} \ge 0$ for $p = q-1$, then it is obvious that $\beta_k \ge 0$ for all k. If not, since $\mu > 1$, there exists $1 \le p_0 \le q-2$ such that

$$\begin{aligned}
0 \;&\le -s_{\pi(n-q+p_0-1)}(x_{\pi(n-q+p_0-1)} - y_{\pi(n-q+p_0-1)})\\
&\quad + (\textstyle\sum_{k=n-q+p_0}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - 1)/(q-p_0)
\end{aligned}$$

and

$$\begin{array}{rl} 0 & > -s_{\pi(n-q+p_0)}(x_{\pi(n-q+p_0)} - y_{\pi(n-q+p_0)}) \\ & \quad + (\sum_{k=n-q+p_0+1}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - 1)/(q - p_0 - 1). \end{array}$$

Hence,

$$\begin{array}{l} s_{\pi(n-q+p_0)}(x_{\pi(n-q+p_0)} - y_{\pi(n-q+p_0)}) \\ \quad + (1 - \sum_{k=n-q+p_0}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}))/(q - p_0) \geq 0. \end{array}$$

Therefore we have that p is equal to p_0. Then $\beta_k \geq 0$ for all k. It is obvious that

$$\sum_{k=0}^{n} \beta_k = 1 \text{ and } x = \sum_{k=0}^{n} \beta_k y^k,$$

where y^k is as defined in Definition 2.1 for $k = 0, 1, \ldots, n$. Thus

$$x \in T_1(y, \pi, s, p).$$

From the above conclusions, the lemma follows immediately.

Theorem 2.3. T_1 is a triangulation of $C^n(m)$.

Proof. From Lemma 2.2 and Definition 2.1, the theorem follows immediately.

We call the triangulation T_1 of $C^n(m)$ the T_1-triangulation.

Let

$$P = \begin{bmatrix} -1 & & & & \\ 1 & -1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ & & & & 1 \end{bmatrix}$$

and $\bar{u}^1 = (1, 0, \ldots, 0)^\top \in R^{n+1}$. Clearly,

$$S^n = m^{-1} P C^n(m) + \{\bar{u}^1\}.$$

Let

$$T_1' = \left\{m^{-1}P\sigma + \left\{\bar{u}^1\right\} \mid \sigma \in T_1\right\}.$$

Then it is obvious that T_1' is a triangulation of the unit simplex S^n. We call the triangulation T_1' of the unit simplex the T_1'-triangulation.

3. THE PIVOT RULES OF T_1-TRIANGULATION

In this section, we give the pivot rules of the T_1-triangulation of $C^n(m)$ and hence also of the T_1'-triangulation of S^n. Let

$$\sigma = T_1(y, \pi, s, p)$$

be a simplex of the T_1-triangulation of $C^n(m)$ with vertices $y^0, y^1, \ldots, y^n$. We want to obtain the parameters of a simplex of the T_1-triangulation, say

$$\bar{\sigma} = T_1(\bar{y}, \bar{\pi}, \bar{s}, \bar{p}),$$

such that all vertices of σ are also vertices of $\bar{\sigma}$ except vertex y^i, unless the facet of σ opposite y^i lies in the boundary of $C^n(m)$. Table 1 describes how $\bar{y}$, $\bar{\pi}$, $\bar{s}$, and $\bar{p}$ are determined from y, π, s, p and i. From this table, it is easy to obtain the vertices of $\bar{\sigma}$, in particular its new vertex.

In this table, $p_1 = \bar{q} - 1$, where $\bar{q}$ is the nonnegative integer such that $\bar{\pi}(k) \in K(\bar{y}, \bar{s})$ for $k = n - \bar{q} + 1, \ldots, n$ and $\bar{\pi}(n - \bar{q}) \notin K(\bar{y}, \bar{s})$, and moreover, if $\bar{y}_{\pi(n-1)} = \bar{y}_{\pi(n)}$ and $\bar{s}_{\pi(n-1)} = \bar{s}_{\pi(n)}$, $p_2 = p_3 = p_4 = p_5 = 0$; otherwise $p_2 = p_3 = p_4 = p_5 = p$.

4. THE COMPARISONS OF TRIANGULATIONS OF THE UNIT SIMPLEX

In [2], it was demonstrated that the D_1-triangulation of R^n is superior to the other well-known triangulations of R^n according to measures of efficiency of triangulations, for example, according to the number of simplices in a unit cube, the diameter, the average directional density, and the surface density. From the definition of the T_1-triangulation, it can be seen that the T_1-triangulation is obtained by combining the D_1-triangulation and the J_1-triangulation, i.e., we triangulate all cubes with one of its vertices belonging to T_1^c according to the D_1-triangulation. Hence the T_1-triangulation

Table 1(1): The Pivot Rules of the T_1-Triangulation

i		p		$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$
0	$q = 1$			y	$s - 2s_{\pi(1)}$ $u^{\pi(1)}$	π	p
$1 \leq i$ $< n$			$y_{\pi(i)} = y_{\pi(i+1)}$ $s_{\pi(i)} = s_{\pi(i+1)}$				
			otherwise	y	s	$(\pi(1), \ldots, \pi(i+1),$ $\pi(i), \ldots, \pi(n))$	p
n				$y + 2s_{\pi(n)}$ $u^{\pi(n)}$	$s - 2s_{\pi(n)}$ $u^{\pi(n)}$	π	p_1
0	$2 \leq q$ $< n - 1$			y	$s - 2s_{\pi(1)}$ $u^{\pi(1)}$	π	p
	$2 \leq q$ $= n - 1$	0	$\pi(1) \in J(y)$	y	$s - 2s_{\pi(1)}$ $u^{\pi(1)}$	π	p
			$\pi(1) \in I^+(y)$ $\cup I^-(y)$	y	$s - 2s_{\pi(1)}$ $u^{\pi(1)}$	π	p
		$p \geq 1$	$\pi(1) \in J(y)$	y	$s - 2s_{\pi(1)}$ $u^{\pi(1)}$	π	p
			$\pi(1) \in I^+(y)$ $\cup I^-(y)$	y	$s - 2s_{\pi(1)}$ $u^{\pi(1)}$	π	$p + 1$
$1 \leq i$ $\leq n - q$ -2	$2 \leq q$ $< n$		$y_{\pi(i)} = y_{\pi(i+1)}$ $s_{\pi(i)} = s_{\pi(i+1)}$				
			otherwise	y	s	$(\pi(1), \ldots, \pi(i+1),$ $\pi(i), \ldots, \pi(n))$	p

Table 1(2): The Pivot Rules of the T_1-Triangulation

i		p		$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$
$1 \leq i = n-q-1$	$2 \leq q < n$	0	$\pi(i) \in K(y,s)$	y	s	$(\pi(1),\ldots,\pi(i+1),\pi(i),\ldots,\pi(n))$	p
			$y_{\pi(i)} = y_{\pi(i+1)}$ $s_{\pi(i)} = s_{\pi(i+1)}$				
			otherwise	y	s	$(\pi(1),\ldots,\pi(i+1),\pi(i),\ldots,\pi(n))$	p
		$p \geq 1$	$\pi(i) \in K(y,s)$	y	s	$(\pi(1),\ldots,\pi(i+1),\pi(i),\ldots,\pi(n))$	$p+1$
			$y_{\pi(i)} = y_{\pi(i+1)}$ $s_{\pi(i)} = s_{\pi(i+1)}$				
			otherwise	y	s	$(\pi(i),\ldots,\pi(i+1),\pi(i),\ldots,\pi(n))$	p
$n-q$		0		y	s	π	$p+1$
		1		y	s	π	$p-1$
		$p \geq 2$	$y_{\pi(i)} = y_{\pi(i+1)}$ $s_{\pi(i)} = s_{\pi(i+1)}$				
			otherwise	y	s	$(\pi(1),\ldots,\pi(i+1),\pi(i),\ldots,\pi(n))$	$p-1$
$n-q < i < n-q+p-1$				y	s	$(\pi(1),\ldots,\pi(i+1),\pi(i),\ldots,\pi(n))$	p
$n-q+p-1$		$p \geq 2$		y	s	π	$p-1$

Table 1(3): The Pivot Rules of the T_1-Triangulation

i		p		$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$
$n-q+p-1<i$	$2 \le q < n$	$1 \le p < q-1$		y	s	$(\pi(1),\dots,\pi(n-q+p-1), \pi(i),\pi(n-q+p),\dots, \pi(i-1),\pi(i+1),\dots,\pi(n))$	$p+1$
		$1 \le p = q-1$	$i=n-1$	$y-2s_{\pi(n)} u^{\pi(n)}$	$s-2s_{\pi(n)} u^{\pi(n)}$	π	p_2
			$i=n$	$y-2s_{\pi(n-1)} u^{\pi(n-1)}$	$s-2s_{\pi(n-1)} u^{\pi(n-1)}$	$(\pi(1),\dots,\pi(n),\pi(n-1))$	p_3
$n-q<i$		0	$y_{\pi(i)}=y_{\pi(n-q)}$ $s_{\pi(i)}=s_{\pi(n-q)}$				
			otherwise	y	s	$(\pi(1),\dots,\pi(n-q-1), \pi(i),\pi(n-q),\dots,\pi(i-1), \pi(i+1),\dots,\pi(n))$	p
		0		y	s	π	$p+1$
		1		y	s	π	$p-1$
0		$2 \le p$	$\pi(1) \in I^+(y) \cup I^-(y)$	y	$s-2s_{\pi(1)} u^{\pi(1)}$	π	$p-1$
			$\pi(1) \in I(y)$	y	$s-2s_{\pi(1)} u^{\pi(1)}$	π	p
		0	$\pi(i) \in I^+(y) \cup I^-(y)$	y	$s-2s_{\pi(i)} u^{\pi(i)}$	$(\pi(i),\pi(1),\dots,\pi(i-1) \pi(i+1),\dots,\pi(n))$	p
			$\pi(i) \in I(y)$	y	$s-2s_{\pi(i)} u^{\pi(i)}$	π	p
$1 \le i$	$q=n$	$i<p-1$		y	s	$(\pi(1),\dots,\pi(i+1), \pi(i),\dots,\pi(n))$	p
		$i=p-1$		y	s	π	$p-1$
		$i+1>p$ $1 \le p < n-1$		y	s	$(\pi(1),\dots,\pi(p-1),\pi(i), \pi(p),\dots,\pi(i-1), \pi(i+1),\dots,\pi(n))$	$p+1$
		$n-1$	$i=n-1$	$y+2s_{\pi(n)} u^{\pi(n)}$	$s-2s_{\pi(n)} u^{\pi(n)}$	π	p_4
			$i=n$	$y+2s_{\pi(n-1)} u^{\pi(n-1)}$	$s-2s_{\pi(n-1)} u^{\pi(n-1)}$	$(\pi(1),\dots,\pi(n),\pi(n-1))$	p_5

is superior to the triangulations of $C^n(m)$ that are obtained from the K_1-triangulation or the J_1-triangulation. Therefore, the T_1'-triangulation is superior to the well-known triangulations of the unit simplex.

References

[1] Allgower, E.L. and Georg, K.(1980). Simplicial and continuation methods for approximating fixed points and solutions to systems of equations, SIAM Review 22, 28-85.

[2] Dang, C.(1989).The D_1-triangulation of R^n for simplicial algorithms for computing solutions of nonlinear equations. Discussion Paper 8928, Center for Economic Research, Tilburg University, Tilburg, The Netherlands. To appear in Mathematics of Operations Research.

[3] Doup, T.M.(1988). Simplicial Algorithms on The Simplotope. Lecture Notes on Economics and Mathematical Systems, Springer-Verlag, Berlin.

[4] Doup, T.M., van der Laan, G. and Talman, A.J.J.(1985).The $(2^{n+1}-2)$-ray algorithm: a new simplicial algorithm to compute economic equilibria. Mathematical Programming 39, 241-252.

[5] Eaves, B.C.(1984). A Course in Triangulations for Solving Equations with Deformations. Lecture Notes on Economics and Mathematical Systems, Springer-Verlag, Berlin.

[6] Kojima, M. and Yamamoto, Y.(1984). A unified approach to the implementation of several restart fixed point algorithms and a new variable dimension algorithm. Mathematical Programming 28, 288-328.

[7] van der Laan, G. and Talman, A.J.J.(1979). A restart algorithm for computing fixed points without an extra dimension. Mathematical Programming 17, 74-84.

[8] van der Laan, G. and Talman, A.J.J.(1981). A class of simplicial restart fixed point algorithms without an extra dimension. Mathematical Programming 20, 33-48.

[9] Scarf, H. and Hansen, T.(1973). The Computaton of Economic Equilibria. Yale University Press, New Haven.

[10] Talman, A.J.J. and Yamamoto, Y.(1989). A simplicial algorithm for stationary point problems on polytopes. Mathematics of Operations Research 14, 383-399.

[11] Todd, M.J.(1976). The Computation of Fixed Points and Applications. Lecture Notes on Economics and Mathematical Systems, Springer-Verlag, Berlin.

[12] Todd, M.J.(1976). On triangulations for computing fixed points. Mathematical Programming 10, 322-346.

[13] Wright, A.H.(1981). The octahedral algorithm, a new simplicial fixed point algorithm. Mathematical Programming 21, 47-69.

[14] Yamamoto, Y.(1983). A new variable dimension algorithm for the fixed point problem. Mathematical Programming 25, 329-342.

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